Modelling Emergence of the Interbank Networks

Grzegorz Halaj and Christoffer Kok

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GRZEGORZ HAŁAJ* and CHRISTOFFER KOK
European Central Bank, Kaiserstrasse 29, D-60311 Frankfurt am Main, Germany

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Interbank contagion has become a buzzword in the aftermath of the financial crisis that led to a series of shocks to the interbank market and to periods of pronounced market disruptions. However, little is known about how interbank networks are formed and about their sensitivity to changes in key bank parameters (for example, induced by common exogenous shocks or by regulatory initiatives). This paper aims to shed light on these issues by modelling endogenously the formation of interbank networks, which in turn allows for checking the sensitivity of interbank network structures and hence, their underlying contagion risk to changes in market-driven parameters as well as to changes in regulatory measures such as large exposures limits. The sequential network formation mechanism presented in the paper is based on a portfolio optimization model, whereby banks allocate their interbank exposures while balancing the return and risk of counterparty default risk and the placements are accepted taking into account funding diversification benefits. The model offers some interesting insights into how key parameters may affect interbank network structures and can be a valuable tool for analysing the impact of various regulatory policy measures relating to banks’ incentives to operate in the interbank market.

Keywords: Interbank network; Financial contagion; Counterparty risk; Financial regulation

1. Introduction

The interbank market was one of the main victims of the financial crisis erupting in 2007. The crisis led to a general loss of trust among market participants and resulted in severe interbank market disruptions and even to periodic freezes of certain market segments. Moreover, failures of some key market players triggered concerns about risks of interbank contagion, whereby even small initial shocks could have potentially detrimental effects on the overall system. As a result of these concerns and also reflecting a broader aim of making the financial sector more resilient, in recent years, financial regulators have introduced various measures that aim at mitigating (and better reflecting) the risks inherent through the bilateral links between banks in the interbank network. These international reform initiatives range inter alia from limits on large counterparty exposures (to contain concentration risk) and on imposing additional requirements on the risk weights banks have to apply on exposures to other financial institutions.† While it seems plausible that these initiatives should help alleviate contagion risks in the interbank market, there is still only a little research aiming to quantify and understand the effects of these reforms on network structures and the contagion risk that might emerge from these structures.

Against this background, this paper aims to help fill this gap in the literature by improving our understanding of risks stemming from bank interconnectedness and how specific regulatory measures can affect interbank network structures and hence contagion risk. When trying to assess how different policy measures are likely to impact on interbank network formation, it will be crucial to also take into account how banks could be expected to react to these measures. For this reason, the starting point of the analysis presented in this paper is to establish a setting whereby network structures emerge on the basis of banks’ endogenous reactions to changes in the

*Corresponding author. Email: grzegorz.halaj@ecb.int

†While measures such as large exposure limits have been used for microprudential purposes for several years already, following the financial crisis more emphasis was placed on using regulatory instruments for macroprudential purposes with inter alia the objective of mitigating contagion risk. Specifically, in a European context, for macroprudential purposes, large exposure limits and the possibility to set higher risk weights on intrafinancial sector exposures were introduced in para.458 of the EU Capital Requirements Regulation, CRR.
environment, affecting their optimal asset and liability mix (and hence also their decision to lend in the interbank market).

For this purpose, the paper presents a model to derive interbank networks that are determined by certain characteristics of banks’ balance sheets, the structure of which is assumed to be an outcome of banks’ risk-adjusted return optimization of their assets and liabilities. The model of bank balance sheet optimization is combined with the random network generation technique presented in Halaj and Kok (2013b). This allows us to study the endogenous network formation based on optimizing bank behaviour.

The model can thus help to understand the foundations of the topology of the interbank network. It furthermore provides a tool for analysing the sensitivity of the interbank structures to the heterogeneity of banks (in terms of size of balance sheet, capital position, general profitability of non-interbank assets and counterparty credit risk) and to changes of market and bank-specific risk parameters. Such parameter changes could, for example, be due to regulatory policy actions (for example, pertaining to capital buffers as well as the size and diversity of interbank exposures) aiming at mitigating systemic risk within the interbank system. The framework developed in this paper can therefore be used to conduct a normative analysis of macro- and microprudential policies geared towards more resilient interbank market structures.

The paper is related to research on network formation which was only recently pursued in finance. Understanding the emergence process of the interbank networks can be critical to control and mitigate these risks. Endogenous networks (and their dynamics) are difficult problems since the behaviour of the agents (banks in particular) is very complex. In other areas of social studies, the network formation was addressed by means of network game techniques (Jackson and Wolinsky 1996). In financial networks, researchers also applied recently game-theoretical tools (Cohen-Cole et al. 2011, Acemoglu et al. 2013, Bluhm et al. 2013) or portfolio optimization (Georg 2011). For instance, Acemoglu et al. (2013) show that the equilibrium networks generated via information-based social learning processes can be socially inefficient since financial agents ‘do not internalize the consequences of their actions on the rest of the network’. In Cohen-Cole et al. (2011) banks respond optimally to shocks to incentives to lend. Moreover, Castiglionesi and Lavarro (2011) presented a model with endogenous network formation in a setting with microfounded banking behaviour. These advances notwithstanding, owing to the complexity of the equilibrium-based studies of network formation, agent-based modelling of financial networks is one promising avenue that can be followed (Markose 2012, Grasselli 2013).

This paper adds to this strand of the literature by taking a model of portfolio optimizing banks to a firm-level data-set of European banks, which in turn allows us to study within an endogenous network setting the impact of plausible internal limit systems based on economic capital calculations of credit valuation adjustments (CVA) accounting for counterparty credit risk (Deloitte and Partners 2013) and various regulatory policy measures on interbank contagion risk. Apart from the asset-liability optimizing behaviour that we impose on the agents (i.e. the banks), our network formation model also incorporates sequential game-theoretical elements. If the portfolio optimization of interbank investment and interbank funding does not lead to a full matching of interbank assets and liabilities, banks will engage in a bargaining game while taking into account deviations in their optimal levels of diversification of investment and funding risks (see e.g. Rochet and Tirole 1996). The sequence of portfolio optimization and matching games is repeated until the full allocation of interbank assets at the aggregate level has been reached. The outlined mechanism is also related to studies on matching in the loan market (see e.g. Fox 2010, Chen and Song 2013). Furthermore, to further reduce mismatches between banks’ funding needs and the available interbank credit emerging from the portfolio optimization choices, we introduce an interbank loan pricing mechanism that is related to models of money market price formation (see e.g. Hamilton 1998, Ewerhart et al. 2004, Einschmidt and Tapping 2009). Importantly, as argued by Afonso et al. (2011), such pricing mechanisms can be expected to be more sensitive to borrower characteristics (and risks) during periods of stress. The model presented here would be able to account for such effects.

The paper is structured as follows: Section 2 presents the model of network formation under optimizing bank behaviour. In Section 3, some topology results from the network simulations presented, while in Section 4, it is illustrated how the model can be applied for studying various macroprudential policy measures. Section 5 concludes.

## 2. Model

### 2.1. Outline

The interbank network described in this paper is an outcome of a sequential game played by banks trying to invest on the interbank market and to borrow interbank funding. Banks optimize their interbank assets taking into account risk and regulatory constraints as well as the demand for the interbank funding and propose their preferred portfolio allocation. For what concerns the funding side banks define their most acceptable structure of funding sources with the objective to limit refinancing (rollover) risk. Banks meet in a bargaining game in which the supply and demand for interbank lending is determined. In order to account for the quite complex aspects of the interbank market formation, we propose a sequential optimization process, each step of which consists of four distinctive rounds (see the block scheme in figure 1).

There are three main general assumptions of the model:
Figure 1. The sequential four-round procedure of the interbank formation.

1. **Banks know their aggregate interbank lending and borrowing as well as those of other banks in the system. It is public information for all the banks in the sample.**
2. **Banks optimize the structure of their interbank assets, i.e. their allocation across counterparties.**
3. **Banks prefer diversified funding sources in terms of rollover risk (i.e. liquidity risk related to the replacement of the maturing interbank deposits).**

In the **first round**, banks specify the preferred allocation of interbank assets by maximizing the risk-adjusted return from the interbank portfolio. In this optimization process, each bank first draws a sample of banks according to a pre-defined probability that a bank is related to another bank. The probability map was developed by Halaj and Kok (2013b) using the geographical breakdown of banks’ exposures disclosed during the EBA 2011 capital exercise. Second, they make offers of interbank placements at a current market rate trying to maximize the return adjusted by investment risk taking into account:

- expected interest income;
- risk related to interest rate volatility and potential default of counterparties, and correlation among risks;
- internal risk limits for capital allocated to the interbank portfolio, based on the credit valuation adjustment (CVA) concept† and regulatory constraints in form of the large exposure limits specifying the maximum size of an exposure in relation to the capital base;
- exogenous volume of total interbank lending.

Notably, the structure rather than the aggregate volume of lending is optimized. The aggregate interbank lending and borrowing of banks in the model is exogenous, e.g. an outcome of a preceding ALM process which we do not model.

Obviously, the recipients of the interbank funding can have their own preferences regarding funding sources. Therefore, in the **second round** of the model, after the individual banks’ optimization of interbank assets, banks calculate their optimal funding structure, choosing among banks that offered funding in the first round. They decide about the preferred structure based on the funding risk of the resulting interbank funding portfolios and assumed preferences for diversifying the funding risk.

The assumption of funding diversification preferences may seem somewhat at odds with what is typically observed in real data which suggests that banks in general use only a small number of funding counterparties (see e.g. Craig and von Peter (2010), Fricke and Lux (2012), Langfield et al. (2014), van Lelyveld and in’t Veld (2012) and Abbassi et al. (2013)). However, our approach accounts for this stylized fact in the sense that the probability map already limits the number of banks from which funding are likely to be sought. Hence, in practice our modelling framework is probable to result in banks selecting only a few counterparties from which to obtain funding; in line with empirical facts.

The offers of interbank placements may diverge from the funding needs of the other side of the interbank market. In the **third round**, we therefore assume that pairs of banks negotiate

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†This CVA element is not to be mistaken with the CVA capital charge on changes in the credit spread of counterparties on OTC derivatives transactions. However, the line of calculation is similar. Some banks use CVA internally to render exposure limits sensitive to the counterparty risk in a consistent, model-based way (Deloitte and Partners 2013).
the ultimate volume of the interbank deposit. We model these negotiations by means of a bargaining game in which banks may be more or less willing (or sensitive from an utility perspective) to deviate from their optimization-based preferred asset-liability structures. Notably, also at this round banks take into account their risk and budget constraints.

Since interbank asset and interbank funding optimization followed by the game may not result in full allocation of the pre-defined interbank assets and in satisfaction of all the interbank funding needs the prices on the interbank market may be adjusted. In the fourth round, banks with an open funding gap are assumed to propose a new interest rate for the new interbank investors depending on the relative size of the gap to their total interbank funding needs. Implicitly, we do not model the role of the central bank which normally stands ready to provide liquidity. For instance, the framework can be extended to account for the central bank liquidity provision to banks that cannot cover their funding gaps after several steps of the algorithm and after substantially adjusting their offered interbank interest rates.

The four consecutive rounds are repeated with a new drawing of banks to be included into sub-samples of banks with which each bank prefers to trade. Consequently, each bank enlarges the group of banks considered to be their counterparties on the interbank market and proposes a new preferred structure of the interbank assets and liabilities for the unallocated part in the previous step. In this way, the interbank assets and liabilities are incrementally allocated among banks.

Modelling the network formation process in sequential terms is obviously somewhat stylized as in reality banks are likely to conduct many of the steps described here in a simultaneous rather than sequential fashion. However, the step-by-step approach is a convenient way of presenting the complex mechanisms that determine the formation of interbank linkages.

The following subsections describe in detail how the endogenous networks are derived. Some important notation is put into the footnote.†

### 2.2. Banks

First, a description of banks’ balance sheet structures, interbank assets and liabilities in particular is warranted. It is supposed that there are \(N\) banks in the system. Each institution \(i\) aims to invest \(\alpha_i\) volume of interbank assets and collect \(\lambda_i\) of interbank liabilities. These pre-defined volumes are dependent on various exogenous parameters. For instance, individual banks’ aggregate interbank lending and borrowing can be an outcome of asset and liability modelling (ALM).‡ The interest rates paid by interbank deposits depend on:

- some reference market interest rates \(r^m\) (e.g. the three month offered interbank rate in country \(m\));
- a credit risk spread \(s_i\) reflecting the credit risk of a given bank \(i\);
- a liquidity premium \(q_i\) referring to the general market liquidity conditions and bank \(i\)’s access to the interbank market§;
- loss given default (LGD) related to the exposure, denoted \(\lambda\).

The LGD is assumed to be equal for all banks and exposures and amounting to 40%. We do not model maturity structure of the interbank assets and liabilities in the current setting, i.e. all interbank assets and liabilities have the same maturity.

The credit spread \(s_i\) is translated into a bank-specific interest rate paid by bank \(i\) to its interbank creditors—\(r_i\). It is based on the notion of equivalence of the expected returns from interbank investment to a specific bank and from investing into the reference rate \(r^m\):

\[
r^m + q_i \equiv r_i p_i \lambda + (1 - p_i) r_i, \tag{1}
\]

where \(p_i\) denotes marginal probability of default on the interbank placement extended to bank \(i\) and is calculated as

\[
p_i = \frac{s_i}{\lambda}.
\]

Interest rate \(r_i\) can be interpreted as a rate that realizes the expected return of \(r^m\), given the default risk captured by the spread \(s_i\).¶ We use a very basic approximation of the default probability \(p_i\) derived from the spread \(s_i\), but still we are able to gauge differences in default risk among bank and the definition of \(p_i\) is not key in developing the modelling framework for endogenous interbank networks.

Moreover, the cost—or a return from the interbank placement perspective—is risky. The riskiness is described by a vector \(\sigma: = [\sigma_1, \ldots, \sigma_N]\top\) of standard deviations of historical (computed) rates \(r_i\) and correlation matrix \(Q\) of these rates calculated from equation 1 taking into account time series of interbank rates and CDS spreads.§ The riskiness stems from the volatility of market rates and variability of default probabilities. Likewise, correlation is related to:

- the common reference market rate for banks-debtors in one country or co-movement of reference rates between countries which the cost of interbank funding is indexed to;
- the correlation of banks’ default risk.††

Banks are also characterized by several other parameters not related to the interbank market but important in our framework from the risk absorption capacity perspective:

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**†Notation:** \(\mathbb{N}\) stands for set \(\{1, 2, \ldots, N\}\), ‘\(\cdot\)’ denotes entry-wise multiplication, i.e. \(\{x_1, \ldots, x_N\} \times \{y_1, \ldots, y_N\} = \{x_1 y_1, \ldots, x_N y_N\}\).

**‡** is transposition operator and—for matrix \(X\)—\(X_{ij}\) denotes \(j\)th column of \(X\) and \(X_{ji}\) denotes \(i\)th row of \(X\). \(\#C\)—number of elements in a set \(C\). \(I_A\) denotes indicator function of a set \(A\).

**§** Georg (2011) or Halaj (2013) developed frameworks based on the portfolio theory to optimize the structure of investments and funding sources that could be followed.

**¶** As the liquidity premium is not central to the questions we are trying to address in this paper, we assume for simplicity that \(q = 0\) throughout. An extension to a non-zero liquidity premium would however be straightforward.

**‖** Currency risk related to the cross-border lending between countries with different currencies is not addressed in the model.

**§** Other measures can be applied, e.g. VaR-based, reflecting the tail risks or some multiples of standard deviation (two-, three-times standard deviation).

**††** Reason: banks operate on similar markets, have portfolios of clients whose credit quality depends on similar factors and their capital base is similarly eroded by the deteriorating market conditions, etc.
capital \( e_i \); and capital allocated to the interbank exposures \( e_i^j \) (e.g. economic capital budgeted for treasury management of the liquidity desk);

- risk weighted assets \( RWA_i \) — similarly, \( RWA_i^j \) risk-weighted assets calculated for the interbank exposures with corresponding risk weights \( o_j \). This may depend on the composition of the portfolio, i.e. exposure to risk of different counterparts.

All the aforementioned balance sheet parameters are used in the following subsections to define banks’ optimal investment and funding programmes.

2.3. First round—optimization of interbank assets

Each bank is assumed to construct its optimal composition of the interbank portfolio given market parameters, risk tolerance, diversification needs (also of a regulatory nature) and capital constraints (risk constraints including the credit valuation adjustment (CVA)).

2.3.1. Prerequisites. Let \( L_{ij} \) denote an interbank placement of bank \( j \) in bank \( i \). Bank risk aversion is measured by \( \kappa \geq 0 \). CVA is assumed to impact the economic capital and, consequently, the potential for interbank lending.\(^\dagger\) For simplicity, we assume that an interbank exposure of volume \( L_{ij} \) requires \( \gamma_j L_{ij} \) to be deducted from capital \( e_j^i \), for \( \gamma_j \) being bank specific CVA factor, to account for the market-based assessment of the credit risk related with bank \( i \). A possible way to calculate CVA is presented in the appendix.

Banks are assumed to trade most likely with banks with which they have an established customer relationship. This is proposed to be captured by banks’ geographical proximity as well as the international profile of the bank. It is assumed that banks are more likely to trade with each other if they operate on the same market. The probability map \( (P) \) of interbank linkages, introduced by Halaj and Kork (2013b), and calculated based on the banks’ geographical breakdown of exposures, is used as sample banks with which a given bank intends to trade. The maturity is standard and common across the market and the rate is determined by the reference rate and the credit quality of the borrower (see identity 1).

2.3.2. Procedure. Since the formation of the interbank network is modelled in a sequential way, we set the initial values at the steps \( k = 1, 2, 3... \) and of a structure of the interbank network, i.e. for \( k = 0 \)

\[
\begin{align*}
\bar{L}^0 & = l, \\
\bar{A}^0 & = a, \\
\bar{L}^0 & = 0^{N \times N}
\end{align*}
\]

Vectors \( \bar{A}^k \) and \( \bar{L}^k \) denote banks’ aggregate interbank lending and borrowing which is still not allocated among banks before step \( k \). A matrix \( \bar{L}^k \) denotes the structure of linkages on the interbank market created up to the step \( k \) of the algorithm. Additionally, for notational convenience we denote \( B_j^k = \emptyset \) as the initial empty set of banks in which a given bank \( j \) intends to invest.

At step \( k \), bank \( j \) draws a sample of banks \( B_j^k \subset \mathbb{N}/(j) \). More specifically, each counterparty \( i \) of the bank \( j \) is accepted with probability \( P_{ij} \).\(^\ddagger\) Banks from the set \( B_j^k \) are assumed to enlarge the set of investment opportunities of bank \( j \), i.e. \( B_j^{k+1} = B_j^{k-1} \cup B_j^k \). At step \( k \), the bank considers (optimally) extending interbank placements to banks \( B_j^k \).

Bank \( j \) is assumed to maximize the following risk-adjusted return from the interbank investment:

\[
J(\bar{L}_i^1, \ldots, \bar{L}_N^k) = \sum_{i,j} r^{ij} \bar{L}_{ij}^k - \kappa_j (\sigma * \bar{L}_j^k)^\top Q (\sigma * \bar{L}_j^k),
\]

where \( r^{ij} \equiv r \) and rates \( r^{ij} \) in steps \( k \geq 2 \) of the endogenous network algorithm can vary according to adjustments related to the funding needs of banks that have problems with finding enough interbank funding sources (see Section 2.6). The vector of risk measures \( \sigma \) was defined in Section 2.2. The interest rates \( r^{ij} \) paid by the interbank deposits are the transaction rates defined by equation 1 and the risk—both related to market interest rate risk and default risk—is captured by the covariance \( (\sigma * \bar{L}_j^k)^\top Q (\sigma * \bar{L}_j^k) \).

Given the drawn sample \( B_j^k \), the set of admissible strategies is

\[
A^k_j := \{ y \in \mathbb{R}_+^N | n \notin B_j^k \Rightarrow y_n = 0 \}
\]

subject to further constraints related to risk and regulations. \( A^k_j \) can be interpreted as set of bank \( j \’\)s actions allowing for investing only in the drawn subsample. Obviously, starting from a different seed the sampling may cover any configurations of banks which are allowed by the probability map \( P \).

The maximum value of the functional (2) always exists; however, it may not be unique. This may happen if there are banks with the same characteristics of return and risk. Theoretically it is highly unlikely; however, in practice, we use peers’ parameters for banks with unavailable individual data on interbank interest rates and credit default spreads. Having two identical banks with respect to return and risk parameters means that other market participants can be indifferent to which of them to lend. In our setting, only the size of banks and their customer relationship \( P \) matter. Therefore, we calculate the theoretically optimal breakdown of the interbank placements taking into account a random representative for a group of identical banks and then average out the results.

In the baseline setting of the endogenous networks, we do not restrict the size of exposures a bank is allowed to hold against another bank. However, in practice, banks are constrained by so-called ‘large exposure limits’ (LE).\(^\S\) To account for such regulations, we impose one additional condition:

- each exposure should not exceed \( \chi > 0 \) fraction of the total regulatory capital.

\(^\dagger\)BCBS (2011) stipulates rules to account for the counterparty risk in the regulatory capital. From that viewpoint and for consistency, \( e_j^i \) can also be treated as regulatory capital.

\(^\ddagger\)It can be thought of as a trader of bank \( j \) calling the counterparties randomly, but potentially with higher chance of selecting banks more closely related in trading with \( j \).

\(^\S\)See Article 111 of Directive 2006/48/EC. LE limits were originally introduced in the EU in 1994 in the context of the Large Exposure Directive.
In the current EU Capital Requirements Directive $\chi$ is assumed to be equal to 0.25. Moreover, there is an additional requirement that the sum of all exposures that (individually) exceed 10 per cent of the capital should not surpass 80 per cent of capital.

The second requirement would introduce a nonlinearity in the set of constraints in our model and we decide not to include it. However, the large exposure limit imposed on the individual interbank placement proves to be a more stringent constraint and its severity can be tuned and tested by shifting it sufficiently below 25%. All in all, our baseline set-up of the model excludes the large exposure limit constraints which are introduced for sensitivity analysis of the network structures.

The maximization of the functional (2) is subject to some feasibility and capital constraints.

(i) budget constraint—$\sum_{i\neq j} L_{ij} = \bar{a}_i^k$ and $L_{ij} = 0$, where—just to remind—$\bar{a}_i^k$ is exogenously determined;

(ii) counterparty’s size constraint—$L_{ij} \leq \bar{L}_i^k$;

(iii) capital constraint—$\sum_{i\neq j} \omega_i (\bar{L}_{ij}^k + L_{ij}^k) \leq \epsilon_i^k - \sum_{i\neq j} y_i (\bar{L}_{ij}^k + L_{ij}^k)$.

In the current EU Capital Requirements Directive $\chi$ is a feasible constraint. § The procedure can be interpreted as the banks’ gradual adjustments the total interbank assets until the network sensibility of the network structures.†

1.1. First round—accepting placements according to available interbank funding

The funding side of the market is assumed to accept placements according to their funding structure preferences, while applying the funding diversification risk criteria.

In order to quantify the funding risk, let us suppose that $X_j$ is a random variable taking values 0 and 1 with probability $p_j$ inferred from the credit default spreads $s_j$ (see 2.2) and $1 - p_j$. Obviously, $p_j$ is also a random variable. For a uniformly distributed $u_j$, on the interval [0, 1], independent of $p_j$ and $u_i$, for $i \neq j$, $X_j$ has the following concise representation:

$$X_j = \mathbb{I}_{(u_j \geq p_j)}$$

The variable $X_j$ represents a rollover risk of a bank accepting funding from bank $j$ due to the default probability of $j$. Let $\hat{D}_X^2$ denote the covariance matrix of $[X_1, \ldots, X_N]$ with the underlying correlation of $X_i$s being matrix $Q^X$. The covariance has a representation in a closed form formula, the derivation of which is presented in appendix.

Each bank $i$ aims at minimizing the funding risk. It is assumed that a default of a creditor results in an inability to roll over funding which means materialization of the funding risk. The risk is measured by the variance of the funding portfolio. For a vector of deposits $[L_{i1}^k, \ldots, L_{iN}^k]$ it is quantified by $F: \mathbb{R}_+^N \to \mathbb{R}$ defined:

$$F \left( L_{i1}^k, \ldots, L_{iN}^k \right) = \kappa^F \left[ L_{i1}^k \ldots L_{iN}^k \right] D_X \left[ L_{i1}^k \ldots L_{iN}^k \right]^\top,$$

where $\kappa^F$ is funding risk aversion parameter.

Banks need to choose the composition of their interbank funding portfolios taking as a constraint the set $\bar{B}_j^k$ of banks that, first considered an option to extend a placement to them and, second the total capacity of their counterparties at step $k$. Formally, the admissible set $A^f_i$ of a bank $i$ is defined as:

$$A^f_i = \left\{ y \in \mathbb{R}_+^N | \exists j \in \bar{B}_j^k \Rightarrow y_j \leq \bar{a}_j^k \text{ and } j \notin \bar{B}_j^k \Rightarrow y_j = 0 \right\}$$

In other words, the non-zero components of vectors belonging to $A^f_i$ can only be those $j$s that satisfy: $\bar{B}_j^k \ni i$, i.e. bank $j$ has drawn bank $i$ to be a candidate of a counterparty for its interbank investment portfolio.

Consequently, minimization of the funding risk for bank $i$ means solving the following program:
minimize $F(y)$ on $\mathcal{A}_i^F$
subject to
- budget constraint:
  $\sum_j y_j = r_i^b$,
- limit on cost of funding: $(\hat{r}_i^b + L_{i}^k) r \leq r_j^b$. Banks are willing to pay on their interbank funding rates on average $r_j^b$. This internal limit is related to the expected profitability of assets. It is assumed that if the average cost of funding exceeds the limit, the bank’s return on interbank liabilities is negative.

The minimizing vector is denoted $L_i^{f,k}$. The optimization of the funding portfolio is performed by all the banks in the system simultaneously.

The budget constraint may be too stringent simply because of an insufficient supply of the interbank funding following the first round of the optimization process. Analogously to the interbank asset optimization, bank $i$ tries to solve the funding problem with a slightly relaxed budget constraint, i.e. replacing $\hat{r}_i^b$ with $\hat{r}_i^b - \Delta \hat{r}_i^b$, $\hat{r}_i^b - 2\Delta \hat{r}_i^b$, ... until for some $n_i^{r,k} \in \mathbb{N}$, depending on the step $k$, $\hat{r}_i^b - n_i^{r,k} \Delta \hat{r}_i^b$ is a feasible constraint.

The optimization across all the banks gives an alternative interbank matrix $L_i^{f,k}$ taking into account funding needs and risks. The matrix $L_i^{f,k}$ is composed of vectors $L_{ij}^{f,k}$ in the following way:

$$L_i^{f,k} := \begin{bmatrix}
(L_{1}^{f,k})^T \\
\vdots \\
(L_{N}^{f,k})^T
\end{bmatrix}$$

### 2.5. Third round—bargaining game

The interbank structure $L_i^{f,k}$ may be, as is usually the case, different from $L_i^{F,k}$. In those instances, banks may need to somewhat deviate from their optimized interbank asset-liability structure and therefore, enter into negotiations with other banks in a similar situation. In order to address the issue about banks’ willingness to accept a counteroffer to the optimization-based placement, we consider each pair of banks entering a type of bargaining game with utilities (or disutilities) reflecting a possible acceptable deviation from the optimal allocation of portfolios. The game is performed simultaneously by all pairs of banks. The disutility—which is assumed to be of a linear type—is measured by a change of the optimized functional to a change in the exposure between the preferred volumes $L_{ij}^{f,k}$ and $L_{ij}^{F,k}$.

More specifically, the proposed games give one possible solution to the following question: what may happen if at step $k$, bank $j$ offers a placement of $L_{ij}^{f,k}$ in bank $i$ and bank $i$ would optimally fund itself by a deposit $L_{ij}^{F,k}$ from bank $j$, which is substantially different in volume from the offered one? Perhaps the banks would not reject completely the offer since it may be costly to engage in finding a completely new counterparty. By doing that they may encounter risk of failing to timely allocate funds or replenish funding since the interbank market is not granular. Instead, we assume that these two banks would enter negotiations to find a compromising volume. We model this process in a bargaining game framework. Banks have their disutilities to deviate from the optimization-based volumes. The more sensitive their satisfaction is to the changes in the individually optimal volumes, the less willing they are to concede. We assume that each pair of banks play the bargaining game at each step of the sequential problem in isolation taking into account their risk constraints. This is a key assumption bringing the framework to a tractable one.

The key parameters to define the game are the sensitivities of a bank’s optimized functional to a move from the optimum to the second player’s optimum.‡ The asset side bank’s satisfaction is measured by risk-adjusted return at the optimal allocation. The funding side bank’s utility is gauged by the satisfaction is measured by risk-adjusted return at the optimal allocation. Formally,

$$U_{ij}^{a,k}(x) = J \left( L_{ij}^{1,k}, \ldots, L_{ij}^{N,k}, x, L_{i+1,j}^{1,k}, \ldots, L_{iN}^{1,k} \right)$$

Analogously, the utility $U_{ij}^{f,k}$ of the bank borrower $i$ accepting placement $x$ from bank $j$ is measured at the optimal funding volumes from all other banks than $j$: $U_{ij}^{f,k}(x) = -F \left( L_{ij}^{1,k}, \ldots, L_{ij}^{N,k}, x, L_{i+1,j}^{1,k}, \ldots, L_{iN}^{1,k} \right)$.

Obviously, $U_{ij}^{a,k} = U_{ij}^{a,k} \left( L_{ij}^{1,k} \right)$ and $U_{ij}^{f,k} = U_{ij}^{f,k} \left( L_{ij}^{1,k} \right)$ are the measures of satisfaction at the optimum.

If $L_{ij}^{f,k} \neq L_{ij}^{F,k}$ then we define sensitivity measures of satisfaction of player $j$ moving from the individually optimal allocation $L_{ij}^{F,k}$ to $L_{ij}^{F,k}$ and player $i$ changing the funding volume obtained from player $j$ from $L_{ij}^{F,k}$ to $L_{ij}^{F,k}$. We measure the sensitivity by means of the following ratios:

$$s_{ij}^{a,k} = \max \left( \frac{U_{ij}^{a,k} \left( L_{ij}^{1,k} \right) - U_{ij}^{a,k} \left( L_{ij}^{F,k} \right)}{|L_{ij}^{F,k} - L_{ij}^{1,k}|}, 0 \right)$$

$$s_{ij}^{f,k} = \max \left( \frac{U_{ij}^{f,k} \left( L_{ij}^{F,k} \right) - U_{ij}^{f,k} \left( L_{ij}^{F,k} \right)}{|L_{ij}^{F,k} - L_{ij}^{F,k}|}, 0 \right)$$

In this way, we implicitly assume that banks’ dissatisfaction from abandoning the optimization-based investment and funding portfolios is growing linearly along the allocation between $L_{ij}^{F,k}$ and $L_{ij}^{F,k}$. The max operation accounts for the fact that actions $L_{ij}^{F,k}$ and $L_{ij}^{F,k}$ are not globally optimal but rather on a constraint set of strategies.

In the interbank bargaining game at round $k$, banks maximize the utility functional $G_{ij}^{k}$ defined:

‡The monitoring of such limiting values are critical for banks’ income management processes. Typically, limits are implied by budgeting/funding transfer pricing (FTP) systems (see Adam 2008 for definitions and applications). In order to deactivate this option for a bank $i$, $r_j^b$ needs to be set to a very large number.

‡We prefer this definition of sensitivity to an alternative one based on a unit change or derivative to mitigate problems of missing second-order terms. Notably, the optimized functionals in round 1 and 2 are quadratic.
Case 1 $L_{ij}^{k+1} > L_{ij}^{F,k}$ \Rightarrow
\begin{equation}
G_{ij}^{k}(x) = \left[ U_{ij}^{l,k} - s_{ij} \cdot (x - L_{ij}^{F,k}) \right] \times \left[ U_{ij}^{a,k} - s_{ij} \cdot (L_{ij}^{k} - x) \right]
\end{equation}

maximized on $\left[ L_{ij}^{F,k}, L_{ij}^{k} \right]$

Case 2 $L_{ij}^{k+1} < L_{ij}^{F,k}$ \Rightarrow
\begin{equation}
G_{ij}^{k}(x) = \left[ U_{ij}^{l,k} - s_{ij} \cdot (x - L_{ij}^{k+1}) \right] \times \left[ U_{ij}^{a,k} - s_{ij} \cdot (L_{ij}^{k} - x) \right]
\end{equation}

maximized on $\left[ L_{ij}^{k+1}, L_{ij}^{k} \right]$

After basic calculations, the solution can simply be written as: Case 1 the bargaining game equilibrium allocation $L_{ij}^{G,k}$ satisfies:
\begin{equation}
L_{ij}^{G,k} = \max \left( \min \left( x_{ij}^{k+1}, L_{ij}^{F,k} \right), L_{ij}^{k} \right)
\end{equation}
where
\begin{equation}
x_{ij}^{k+1} = \frac{1}{2} \left[ L_{ij}^{F,k} + L_{ij}^{k} + \frac{U_{ij}^{l,k}}{s_{ij}} - \frac{U_{ij}^{a,k}}{s_{ij}} \right]
\end{equation}

Case 2 the bargaining game equilibrium allocation $L_{ij}^{G,k}$ satisfies:
\begin{equation}
L_{ij}^{G,k} = \max \left( \min \left( x_{ij}^{k+1}, L_{ij}^{F,k} \right), L_{ij}^{k} \right)
\end{equation}
where
\begin{equation}
x_{ij}^{k+1} = \frac{1}{2} \left[ L_{ij}^{F,k} + L_{ij}^{k} + \frac{U_{ij}^{l,k}}{s_{ij}} - \frac{U_{ij}^{a,k}}{s_{ij}} \right]
\end{equation}

On aggregate, the outcome of the bargaining game may violate the $\bar{a}_{ij}$ and $\bar{p}_{i}$ constraint, since $\sum_{i} L_{ij}^{G,k} > \bar{a}_{ij}$ for some $j'$ and $\sum_{j} L_{ij}^{G,k} > \bar{p}_{i}$ for some $i'$. Therefore, rows and columns of $L_{ij}^{G,k}$ are transformed proportionately:
\begin{align}
L_{ij}^{G,k} &\rightarrow \tilde{L}_{ij}^{G,k} := L_{ij}^{G,k} \cdot \min \left( 1, \frac{\bar{p}_{i}}{\sum_{j} L_{ij}^{G,k}} \right), \\
L_{ij}^{G,k} &\rightarrow \tilde{L}_{ij}^{G,k} := L_{ij}^{G,k} \cdot \min \left( 1, \frac{\bar{a}_{ij}}{\sum_{i} L_{ij}^{G,k}} \right)
\end{align}

The ultimate matrix of exposures $\tilde{L}_{ij}^{G,k}$ realized in the step $k$ is defined as the element-wise minimum of $\tilde{L}_{ij}^{G,k}$ and $L_{ij}^{G,k}$:
\begin{equation}
\tilde{L}_{ij}^{G,k} = \min \left( \tilde{L}_{ij}^{G,k}, L_{ij}^{G,k} \right)
\end{equation}

The bargaining game implies that the interbank network matrix at the next step $k+1$ is given as
\begin{equation}
L_{ij}^{k+1} := \tilde{L}_{ij}^{k} + \tilde{L}_{ij}^{G,k}
\end{equation}

Since in this way part of unallocated interbank assets before step $k$ is now invested, then the $k+1$ total interbank assets and liabilities are updated in the following way:
\begin{align}
\tilde{p}_{i}^{k+1} &:= \tilde{p}_{i}^{k} + \sum_{j} L_{ij}^{G,k} \\
\tilde{a}_{ij}^{k+1} &:= \tilde{a}_{ij}^{k} + \sum_{i} L_{ij}^{G,k}
\end{align}

### 2.6. Fourth round—price adjustments

Both the individual optimization and the bargaining game at round $k$ may not lead to the full allocation of the interbank assets and still there may be some banks striving for interbank funding. By the construction of the bargaining game, there are no banks with excess funding sources. In order to increase the chance of supplementing the interbank funding in the next step, banks with interbank funding deficiency adjust their offered interest rate. The adjustment depends on the uncovered funding gap. Let us assume that the market is characterized by a price elasticity parameter $\alpha$ which translated the funding position into the new offered price. If at the step $k + 1$ the gap amounts to $g_{i}^{k+1} := l_{i} - \sum_{j} L_{ij}^{k+1}$ then, the offered rate $r_{i}^{k+1} = r_{i}^{k} \exp(\alpha g_{i}^{k+1} / l_{i})$.

### 2.7. Repeated steps

The initially drawn sample of banks $B_{j}^{1}$ may not guarantee a full allocation of interbank assets across the interbank market. There are various reasons for that: some samples may be too small, consisting of banks that are not large enough to accept deposits or not willing to accept all offered deposits given their preferred interbank funding structure. Therefore, at each step the samples are enlarged by randomly drawing additional banks (again with the probability $P$). Each step of the sequence composed of the optimization of the interbank assets (see Section 2.3) followed by the selection of the preferred interbank funding structure (see Section 2.4), bargaining game (see Section 2.5) and price adjustment of interbank deposits (see Section 2.6) is repeated for the unallocated assets $\bar{a}_{ij}$ and liabilities $\bar{p}_{i}$ until no more placements of significant volume are added to the network. The sequence of rounds 1–4 is terminated when the contribution of matrix $L_{ij}^{G,k}$ is marginal comparing with the interbank network $\tilde{L}_{ij}^{k}$. This is verified by setting an accuracy threshold $\epsilon << 1$ and comparing it with
\begin{equation}
\max_{i} \frac{\sum_{j} \tilde{L}_{ij}^{G,k}}{l_{i}} \quad \text{and} \quad \max_{j} \frac{\sum_{i} \tilde{L}_{ij}^{G,k}}{a_{ij}}
\end{equation}

In most of the applications it takes about 10 steps to allocate more than 90% of the predefined interbank assets $a$. A thorough analysis of convergence is presented in Section 3.2.

### 3. Results

#### 3.1. Data

The model was applied to the EU banking system. The data-set regarding balance sheet structures of banks was the same as the one applied by Halaj and Kok (2013b). Briefly, it contains:

- A sample of 80 large EU banks, predominantly from euro area countries. The sample is a subset of the banks considered in the EBA 2011 EU-wide stress test exercise and the data-set partly relies on public bank level

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\[1\]So far, we do not have a good calibration of $\alpha$ at hand and in the applications illustrated in Section 4 we assume $\alpha = 0$. Nevertheless, sensitivity analysis of the model with respect to $\alpha$ is presented in Section 3.
disclosures from that exercise. The sample excludes two smaller German banks and nine small Spanish cajas for which either interbank lending or borrowing volumes, or capital figures were unavailable. As the sample contains all the major EU banks, it broadly covers the most important players in the European interbank market.

- Bankscope van Dijk’s (BvD’s) data on individual banks’ balance sheet aggregates of total assets (TAij), interbank borrowing and lending, customer loans (Li), securities holding (Si) and capital position (ei);
- Risk-weighted assets of banks in the sample broken down (if available) by total customer loans, securities and interbank lending. These pieces of information are used to proxy the allocation of capital to the interbank exposures. Assuming the Basel II 20% risk weight (RW) for the interbank lending and calculating the average risk weights for customer loans and securities in the sample, denoted RWli and RWsi, respectively. The allocated capital ei is approximated in the following way:

\[
ed_i = \frac{20\% a_i}{RWli + RWsi - Si}
\]

The averages of risk weight of customer loans and securities instead of the bank by bank weight were necessitated by gaps in the data set with respect to the portfolio breakdown of RWAs;
- The geographical breakdown of banks’ aggregate exposures allow for parameterization of the probability map P.

The straightforward caveat of the approximation of ei is that the averaging of RWli and RWsi across banks may lead to excessively stringent capital constraints for some of the banks. The compromising procedure of replacing of total interbank assets ai with ai = kiΔai accounts for that as well.†

Additionally, CDS spreads (s)—for individual banks if available, otherwise country-specific—and three-month money market rates for EU countries (r) were used to approximate the bank-specific interbank rates and their riskiness measured by the standard deviation of rates. For the estimation of γiLi, i.e. the CVA-based capital ratio adjustment related to banks’ interbank exposures, projected paths of individual bank CDS spreads were applied.‡

The estimation of the correlations Q and QX is followed by the testing of the statistical significance of the entries. Insignificant ones (at the probability level of 5%) are replaced by zeros. Three years of data with monthly frequency are used for the estimation.

3.2. Convergence

The convergence of the proposed procedure has to be proved. In fact, stabilization of the process is obvious—it is a non-decreasing, bounded-from-above process. Conditions under which the process converges to full allocation of interbank assets can easily be defined. A very simple, negative condition is the following:

Let \(\text{sgn}: \mathbb{R}^{N \times N} \rightarrow \{-1, 0, 1\}^{N \times N}\) be a sign function of elements of a given matrix. Let us assign \(l^k_P := \text{sgn}(P) \cdot l\) and \(a^k_P := (a^\top \cdot \text{sgn}(P))^\top\). Then, all the interbank assets and interbank liabilities cannot be fully matched if there exists bank \(i\) such that either \(l_i^k < a_i\) or \(a_i^k < l_i\).

This is a very simple criterion that is easy to verify. It provides information about the inherent inability of bank \(i\) to either place a pre-defined volume of money on the interbank or to find sufficient interbank funding sources because of its low connection to the system. The condition is satisfied in the analysed system.

3.3. Structure of endogenous interbank networks

It is far from obvious what the network resulting from the endogenous mechanism may look like. Some common statistical measures can help in understanding the structure at large. In general, the interbank networks are not complete. On average, bank nodes have a degree (normalized to the total number of potential links) of not more than 0.20, but the dispersion among nodes is substantial with some nodes having a degree of 0.30. The results are shown in figure 2. In just a few steps, the algorithm attains an allocation above 80% and the convergence pace decreases significantly. After 20 steps, almost all paths lie above 95% of the allocation ratio. For practical implementation of our algorithm, apart from setting \(\epsilon\) to 0.1%, in order to reduce the computation time, we choose that \(k_{\text{max}} = 20\) steps as the additional stopping criteria for the algorithm.

†Some sensitivity analysis is provided in Section 3.4, table 1.
‡The projected series of bank individual CDS spreads were kindly provided to us by M. Gross and calculated according to a method developed in Gross and Kok (2013).
§The ‘4%’ was calculated as \(\sum_j (\min(1, a_j^k / l_j) - 1) \times l_j / \sum_j l_j\).
¶We provide the normalized degree measures, i.e. degree numbers divided by the number of potential linkages between \(N\) nodes equal to \(N(N-1)/2\).
times higher for some particular nodes. The sparsity of the network is related to the algorithm of drawing the links between banks from the probability map. The allocation of the majority of the interbank exposures takes place at the initial steps of the endogenous network formation algorithm, i.e. for linkages drawn at steps $k$ equal 1, 2, 3 or 4.

Some studies focus on core/periphery properties which mean that there is a subset of nodes in the system that is fully connected, whereas other nodes are only connected to that subset. There are various algorithms selecting the core that may lead to a fuzzy classification—some nodes are 'almost' core or 'almost' periphery. In case of our endogenous networks, we have not found any significant classification of nodes to the core and periphery (using the Borgatti and Everett (1999) approach). This is probably due to the fact that we capture global, internationally active bank hubs and domestic banks, usually strongly connected subsystems of the interbank networks. Overall, these findings suggest that the endogenous networks algorithm generates interbank structures that are not easy to be classified in a simple way by just a few topological parameters.

A usual approach to get a deeper understanding of the network structure is to compare it with some known, well-studied graphs that possess the same statistical (topological) properties. The best known example of a random graph is Erdős-Rényi model (E-R), constructing an edge between two given nodes with a given probability $p$, independent of all other pairs of nodes. Since we operate with a probability map assigning different probabilities to links between different banks, it is straightforward to imagine that the E-R approximation of
endogenous networks should fail.† A more promising method in terms of accuracy of approximations is generated based on detailed information about degree and clustering of the endogenous networks. An expected degree graph (Chung and Lu 2002) is the first example. In this model, links between nodes \( i \) and \( j \) are drawn with probability \( \frac{\deg_i \deg_j}{\sum_k \deg_k} \), where \( \deg_i \) is a degree of a node \( i \). The second type of potentially useful graphs is a random clustered graph model (Newman 2009). Given a degree sequence of all nodes and a triangle sequence of nodes,‡ the random clustered algorithm chooses linkages uniformly from a joint distribution of possible set of triangles complying with the given degree sequence. In this way, the algorithm potentially has a better control not only of the degree distribution, but also of clustering coefficients which are important indicators of contagion transmission channels.§ The third—very natural in the context of our model—type of graphs for comparison are generated from the simulated networks algorithm of Halaj and Kok (2013b). In short, the simulated network algorithm creates the linkages in the network by randomly picking a pair of banks \( (i, j) \) and accepting (rejecting) a linkage between them with probability \( p_{ij}^{geo} \) (1 – \( p_{ij}^{geo} \) respectively). Therefore, both in endogenous and simulated networks the probability map of connections plays a central role in the model.

The results of the comparison of the endogenous and random graphs are shown in figure 3. Random graphs are constructed in such a way that for a given endogenous interbank network (EIN):

†In order to save space in the paper, we do not report the results of the comparison which only confirm intuition.

‡Triangle degree of a node is a number of triangles containing a given node. Triangle sequence is the sequence of triangle degrees of all the nodes in a graph.

§Squartini et al. (2013) shows how important the triangle structures (so-called, triadic motifs) in a directed graph are for an appropriate measure of contagion risk. They propose an algorithm that tries to preserve the observed structure of triangles in a given network.

Bargigli et al. (2013) gives empirical evidences that random models do not necessarily replicate the triadic structures.
• the expected degree graph is generated using the degree sequence of nodes in EIN;
• the random clustered graph is generated with a sequence of pairs consisting of a degree and triangle degree of all nodes.

We analyse 100 realizations of endogenous networks and 100 corresponding random networks. The generated expected degree networks are almost identical to the endogenous networks with respect to the degree distribution. It is not surprising, given that degree of nodes is the only parameter of the expected degree graph algorithm. However, betweenness centrality, measuring direct and indirect connectivity of a given node with all other nodes in the system, proves to be less consistent. Some nodes of the endogenous networks are substantially more important in terms of centrality. The differences between endogenous and expected degree networks are even more striking for clustering measures gauging the concentration of linkages. The random clustered graphs do not perform better, even though their parameters have more degrees of freedom. The algorithm of random clustered networks preserves the ranking of the nodes in terms of degree measures, but produces graphs with nodes possessing many more links than in the corresponding endogenous networks. The resulting clustering coefficients are in general as well as last but not least, the simulated networks possess similar topological properties to the endogenously generated interbank structures. However, the clustering coefficient of the simulated networks is systematically lower in the generated samples of networks. All in all, the complex topology of the endogenously modelled EU interbank network implies that random graphs may oversimplify their structure. These notwithstanding, random graphs offer a valuable benchmarking tool for understanding the relationship between various topological properties of networks.

3.4. Sensitivity analysis

Before drawing policy conclusions, we assess performance of the model verifying the interbank structures it produces and their sensitivity to key driving parameters, such as correlation ($Q$), loss given default ($\lambda$), investment risk aversion ($\kappa$), funding risk aversion ($\lambda^F$), interbank interest rate elasticity ($\sigma$) and capital allocated to the interbank portfolios ($e^I$).

One particular interbank structure estimated in the developed endogenous network model is presented in figure 4. It is incomplete with the largest banks being the most interconnected nodes, which is in line with findings elsewhere in the literature.‡

The correlation structure should, in theory, influence the concentration and completeness of the interbank network. Notably, from the perspective of the optimal asset structures, high correlation translates into low diversification potential implying that the assets are optimally allocated in just a few banks. However, it is not a priori clear how the network structures may respond to the increasing correlation of the interests paid by the interbank deposits. Figure 5 synthetically shows sensitivities of some basic (and commonly used) topological measures of networks to the decreasing correlation. The systemic importance of some nodes gradually increases as the correlation falls from the estimated one to the state, where there is no dependence between interbank interest rates (and default risk that influences the cost of funding). This is slightly more persistent for the larger banks, whereas some of the small banks react strongly as the correlation becomes close to zero by spreading their interbank assets more widely across the system.

The interbank networks may potentially differ a lot for different degrees of correlation. However, it can be observed that the largest linkages, contributing to the core structure of the interbank graph, are generally stable and that the differences are related mainly to the smaller deposit amounts. This is illustrated in the lower pane of figure 6. We compared the original Jaccard index based on the adjacency matrix with a modified one accounting for the sizes of the exposures. It is observed that the curve produced by the modified index lies above the one obtained from the original one, indicating that the linkages among the largest banks are more stable (i.e. less sensitive to correlation). This notwithstanding, in general correlation appears to have a sizable impact on the structure of the endogenous network formation.

There are several key parameters of the model that were set in an arbitrary way. We address the issue of how sensitive the results are to the chosen specification and present the results in Table 1. The structure of the endogenously derived interbank network is most sensitive to deposit interest rate elasticity and the level of allocated capital to the interbank portfolios. It is most evident for the number of linkages between banks (in- and out-degree measures) and betweenness centrality. In general, however, the structures vary in a limited way and the variability that we do observe tends to have intuitive explanations. For instance, increasing investment risk aversion leads on average to the decrease in the number of linkages as measured by in- and out-degree. Consequently, some nodes in the network become more systemic, which can be measured by Katz and DebtRank indicators detecting nodes hubs that directly and indirectly connect banks in the network.‡

The proportionate transformation $L^G \rightarrow \tilde{L}^G$ is designed to ensure that the budget constraint of total interbank assets $a$ and liabilities $l$ is satisfied after the bargaining game, but may distort the outcomes of the game. We verified in how many instances the proportional adjustment is activated. As shown in figure 7, in more than 80% of cases, no adjustment is required and the remaining part of the distribution is spread uniformly in $[0,1]$ interval of adjustments.

4. Policy issues

On the basis of the network formation modelling approach presented in the previous sections, various pertinent policy questions can be addressed. For example, the approach can be employed to detect the impact of different policy measures

‡For a few representative country-specific studies using real-time overnight transactions data or large exposure data as well as entropy approaches, see e.g. Furfine (2003), Upper and Worms (2004), Boss et al. (2004), van Lelyveld and Liedorp (2006), Soramaki et al. (2007) and Degryse and Nguyen (2007).

‡While going beyond the sensitivity analysis of network formation provided here, Ghoshal et al. (2013) provides interesting insights into how network formation processes can be decomposed and analysed.
on the endogenous emergence of network structures and the contagion risks related to those.

This is particularly useful for macroprudential policy analysis purposes. As described in the following, the model can be used to evaluate the impact on interbank network structures of some regulatory instruments aimed at limiting banks’ risk to counterparty exposures, such as the large exposure limits already embedded in current regulatory frameworks. The tool can therefore be used to help calibrate the optimal configuration of such macroprudential and regulatory instruments.

Apart from the evaluation of relevant regulatory instruments, the endogenous network model can also be employed in macroprudential analysis in a broader sense. Namely, for the assessment of the impact of the materialization of systemic risks on the banking sector while taking into account the dynamic network formation resulting from changes in key real economic and financial variables. In particular, as an adverse shock to the economy would typically result in the deterioration of some counterparties’ creditworthiness (e.g. as reflected in banks’ CDS spreads) and hence, affect the optimal allocation of

Table 1. Sensitivity of networks structures to parameters.

<table>
<thead>
<tr>
<th>Parameter (baseline)</th>
<th>Tested value</th>
<th>Out-deg</th>
<th>In-deg</th>
<th>Bness</th>
<th>Katz</th>
<th>DebtRank</th>
</tr>
</thead>
<tbody>
<tr>
<td>LGD (λ) (0.4)</td>
<td>0.3</td>
<td>(0.1)</td>
<td>(0.4)</td>
<td>(0.0)</td>
<td>(0.3)</td>
<td>(0.1)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>1.8</td>
<td>−0.8</td>
<td>−0.5</td>
<td>−1.4</td>
<td>−0.1</td>
</tr>
<tr>
<td>Investment risk aversion (κ) (0.5)</td>
<td>1.5</td>
<td>−3.0</td>
<td>−1.0</td>
<td>0.2</td>
<td>0.5</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>−3.7</td>
<td>−1.0</td>
<td>−4.2</td>
<td>0.7</td>
<td>1.2</td>
</tr>
<tr>
<td>Funding risk aversion (κF) (0.5)</td>
<td>1.5</td>
<td>1.2</td>
<td>0.1</td>
<td>3.4</td>
<td>5.2</td>
<td>3.8</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>2.3</td>
<td>0.1</td>
<td>5.6</td>
<td>5.6</td>
<td>3.7</td>
</tr>
<tr>
<td>Price elasticity (α) (0.0)</td>
<td>0.5</td>
<td>4.1</td>
<td>5.2</td>
<td>10.3</td>
<td>−0.6</td>
<td>−1.8</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>4.4</td>
<td>−2.3</td>
<td>12.8</td>
<td>3.2</td>
<td>−1.7</td>
</tr>
<tr>
<td>Allocated capital (eI) (eI)</td>
<td>0.9eI</td>
<td>2.6</td>
<td>0.6</td>
<td>10.0</td>
<td>−4.2</td>
<td>−3.7</td>
</tr>
<tr>
<td></td>
<td>1.1eI</td>
<td>5.9</td>
<td>3.4</td>
<td>3.7</td>
<td>−1.6</td>
<td>−2.7</td>
</tr>
</tbody>
</table>

Note: Relative change with respect to baseline value of the tested parameter; standard deviations in brackets.

Source: Own calculations.

Figure 6. Network similarity measures versus the strength of the correlation. Note: x-axis: correlation factor reducing the correlation of risk between banks, e.g. 0.4 means that the applied correlation matrix \( Q_{0.4} \) is such that for all \( i \neq j \)
\[ Q_{0.4}(i, j) = \left( 1 - 0.4 \right) Q(i, j) \]; y-axis: dashed line—Jaccard index, solid line—modified Jaccard index (taking into account the size of the exposures (see Appendix 2)).

Source: Own calculations.
interbank assets and liabilities. In other words, a given adverse scenario is likely to result in a re-formation of linkages in the interbank market and hence, to also affect the interbank contagion risks that ultimately may result from such adverse shocks.

The usage of the endogenous network model from these different policy perspectives is described in the following subsections.

4.1. Credit valuation adjustment

As described in Section 2, the CVA charge on economic capital allocated to interbank assets portfolio, usually applied by banks in their internal risk management systems, impacts the interbank asset structure decision-making process via the capital constraint, whereby banks engaged in lending to riskier counterparties will generally face a comparatively higher capital charge to reflect the default risk of their interbank borrowers.

In order to test the impact of CVA-based capital surcharge (or equivalently the CVA-based add-on to the risk weights) on the network structures, constraint three from Section 2.3 including a sensitivity parameter $c$ for CVA was modified as follows:

$$\sum_{i \neq j} (w_i + c y_i) L_{ij} \leq \epsilon'_j$$

By varying $c$ from 0 to 1, the impact of CVA related to a particular configuration of exposures $L_1, \ldots, L_N$ on the capital base increases gradually. Setting parameter $c$ to 0 reflects no additional capital charges for counterparty risk related to the market perception of the credit risk of exposures against banks, whereas $c = 1$ portrays regime with the market-based credit risk valuation of the interbank exposures impacting the allocated capital to the interbank asset portfolio. Simulations of the interbank network structures for different values of $c$ show that network topology does not change substantially except for some smaller (and weaker) banks that are forced to accept less diversified funding sources (in-degree measure drops for some of them). The reason for that is the shift of banks’ interbank placements to more sound institutions as far as the market perception is concerned.

4.2. Large exposure limits

With the objective of limiting counterparty concentration risk banking regulation (e.g. as stipulated in the Basel standards) imposes limits on the size of exposures banks are allowed to hold against other counterparties. As mentioned in Section 2.3, the current EU standard for large exposure limits amounts to 25% of total regulatory capital.

In this subsection, we test the sensitivity of the network structures to a variation of the 25%-threshold. The results are shown in figure 8 for in-degree and out-degree measures, respectively. We observe that network structures (e.g. in terms of number of links of individual nodes) are relatively stable around and especially above the 25%-threshold. Raising the large exposure limits above the 25%-threshold on average would not seem to alter the network structure in any material way. By contrast, a more stringent approach to large exposure limits (i.e. moving the threshold towards 0%) could trigger substantial changes to the structure of banks’ network connections. Intuitively, as limits on large exposures become more binding, banks will have to reduce on the size of individual exposures and as result spread their interbank business across a wider range of counterparties. With some notable exceptions, this is indeed what we observe as reflected in increasing in- and out-degree measures when moving towards the 0%-threshold.
The impact of the large exposure limit threshold on the systemic importance of banks nodes is not uniform across the system. For instance, it can be measured by the betweenness centrality (see figure 9). For the second, third, fifth and tenth banks in terms of the interbank lending activity, more stringent limits lead to a higher diversification of exposures but lower connectivity of nodes in the network.† It does not automatically translate into a safer system from the contagion perspective. A fully fledged simulation based on a propagation of defaults

†Statistical significance was tested based on the Kolmogorov-Smirnov test of differences in distributions.
following some adverse macroeconomic scenarios (see next Section 4.3) can help to assess the impact of regulatory regimes to the contagion risk.

4.3. Effectiveness of policy instruments in adverse market conditions: stress testing

The proposed approach to model the interbank networks opens many potential ways to study the effectiveness of various policy instruments in curbing contagion risk on the interbank market. We focus on the performance of the large exposure limits and in adverse market conditions.

The assessment is related to the dynamic balance sheet model of Hałaj (2013) that characterizes banks’ asset structures. By optimizing the risk-adjusted profitability, banks decide about the allocation of assets to the interbank portfolio. The optimization involves many economic parameters describing banks’ financial standing and their economic environment. This allows for passing through stress testing scenarios to project the evolution of the assets structures under adverse conditions and then—applying the framework for the endogenous interbank market formation—analyse changes in the topology of the interbank market. The main drivers of the interbank structure would be (i) the projected bank individual total interbank assets and (ii) shifts in the profitability of the interbank investments (or from a different angle—availability of funding sources) resulting from the macrofinancial scenario impact on interest rates and counterparty credit risk. Furthermore, by imposing large exposure limits under different economic and financial conditions, the resilience of various structures to contagion propagation can be studied.

The methodology applied to analyse the impact on network structures under different macroeconomic conditions is as follows:

- We first compute each bank’s total interbank investment and funding needs under a baseline macroeconomic scenario. The framework developed by Hałaj (2013) is used to translate the macroeconomic shock into the restructuring of the bank assets. The outcome of the model is the change of volume of broad balance sheet categories: customer loan portfolio, securities portfolio and interbank lending portfolio. The relative change of the volume of the interbank assets of bank \( j \) is used to scale the volume of the interbank funding of \( j \).
- Second, we construct the interbank network applying the method proposed in Section 2 under baseline scenario parameters and total interbank lending and borrowing in various regimes of Large Exposure limits.
- Finally, we impose an adverse macroeconomic shock to banks’ capital position and subsequently run a contagion model of banks’ defaults.† The clearing payments vector approach is used to measure the contagion effects (see Eisenberg and Noe 2001, Hałaj and Kok 2013b).

Figure 10 illustrates the impact of an adverse scenario on the interbank network structures across different network indicators, as compared to the end-sample starting point. Deviations from the 45 degree line imply that the adverse shocks produce a different network configuration. Overall, it is difficult to gauge any systematic pattern in the responses of banks’ interlinkages to the shocks. At the same time, it is notable that a considerable number of banks do deviate from the initial network characteristics, albeit the direction of the changes goes both ways. Some banks increase their degree of interconnectedness to the network, whereas others decrease it. In other words, as should intuitively be expected, a change in the macrofinancial situation will endogenously lead to a change in network structures. We again used statistical tests of differences in mean (t test), variance (F test) and whole distribution (Kolmogorov-Smirnov test) of network measures (in- and out-degree, Katz indicator, DebtRank and betweenness centrality) to verify changes in the network structures. At significance level of 5%, only a hypothesis about differences in variance of Katz ratio and betweenness cannot be rejected, i.e. a one-sided test indicates that in the adverse scenario, some nodes become more interconnected with the whole system comparing to the baseline case. At the level of 10%, significance in-degree measure shows more connections in the system under baseline scenario (the average number of linkages in the system is higher under the baseline). The outcomes of the statistical tests are intuitive. Under the adverse scenario, capital constraints and credit quality of some banks deteriorate resulting in less potential for them to accept interbank deposits (binding capital constraint) or to be offered a placement (binding credit quality). The optimization mechanism proposed in the framework nicely captures the relationship between macrofinancial conditions and network structures.

Apart from assessing whether adverse shocks impact the network formation process, our model can also be employed to evaluate whether contagion losses under an adverse scenario can be mitigated by adjusting certain regulatory (macroprudential) instruments, such as large exposure limits or for example risk weights on exposures to other financial institutions.

†To insure robustness of the results a couple of adverse scenarios was applied.

![Figure 11. Counterparty credit quality and the impact of LE limits on the losses incurred due to contagion. Note: x-axis: CDS spread (in bps); y-axis: difference of CAR after adverse stress testing shock between LE=15% and LE=25% regime (in pp, positive number means that by lowering LE limit contagion losses rise). No CVA adjustment (i.e. \( \gamma \equiv 0 \)). The size of a circle is proportional to a bank’s total assets. Source: Own calculations.](image-url)
For example, figure 11 illustrates the impact of having different LE limit thresholds in the context of an adverse shock. Specifically, the y-axis illustrates the difference between networks formed under a 15% LE limit and under the standard 25% LE limit in terms of the capital loss following an adverse shock. A positive value implies that contagion losses rise when lowering the LE limit. On the x-axis, we plot the banks according to the size of their riskiness (measured in terms of their CDS spreads). It is observed that more stringent LE limits overall tend to lower contagion risk. Interestingly, this effect is especially pronounced for the group of banks perceived (by the markets) to be the soundest. In other words, the forced reduction of counterparty concentration risk that would be implied by a lowering of the LE limits would seem to particularly benefit the safest part of the banking system, whereas the more vulnerable segments are found to be less affected by changes in the LE limits.

5. Conclusions

In this paper, we have tried to deviate from the standard, mechanistic cascading mechanism employed in traditional interbank contagion models. Instead, we have developed an agent-based model that is able to account for banks’ dynamic, endogenous responses both in the formation process of the interbank network and in response to contagious shocks. Apart from analysing network formation processes and the implications for interbank contagion risks in a setting where banks are dynamically optimizing their actions, our model approach can also be used to assess the impact of different regulatory and macroprudential policy instruments on the structure of interbank networks and their implied contagion risk. In this light, we presented a few policy experiments related to the effects of large exposure limits on counterparty credit risks. These macroprudential policy instruments were analysed both in the context of network formation and in terms of their ability to mitigate interbank contagion risks under adverse circumstances.

All in all, while the reported results obviously hinge on the specific characteristics of the banks included in the network system and on the specific adverse scenarios considered, the overriding conclusion from these policy experiments is that macroprudential policies can make a significant difference through their impact on the network formation and ultimately on the risk of interbank contagion to adverse shocks. From this perspective, the modelling approach presented in this paper can be employed for conducting impact assessments of selected macroprudential policy instruments and this way helps to inform the calibration of such tools.

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References


Jackson, M.O. and Wolinsky, A., A strategic model of social and industry and region. Notably, the formula can be used by IRB banks and then the main difficulty of its implementation is finding appropriate profile for EE (in particular maturity τN). The standardized approach banks are required to apply a simplified formula for which the profile of EE does not matter, but still the maturity has to be imposed (at least the contractual of the analysed EAD).

We assume that the netted exposures of a given bank are constant within a given horizon τN. It means that EE is constant, EE = EE* and the vector of proportionality parameters γ used in the definition of the capital constraints have entries defined as:

\[
\gamma_i = \frac{CV\lambda_i - EE^*}{EE + \lambda_i} \times \frac{dn - dn - 1}{2},
\]

Appendix 2. Modified Jaccard index—similarity of networks

In order to measure similarities between networks structures, one can apply the so-called Jaccard index. It counts the number of linkages that appear in the two compared networks and relates it to the total number of linkages in both graphs. It, therefore, neglects the weight of the links, i.e. the importance of the overlapping linkages relative to the one uniquely present in one of the networks. We propose a slight modification to the index accounting for that deficiency. For the two given networks N1 and N2 described by weighted adjacency matrices M1 and M2, span on the same set of N nodes, let us define 3 sets:

\[
M12 = \{(i, j) \in N1 \times N2 | (i, j) \in N1 \wedge (i, j) \in N2\}
\]

\[
M10 = \{(i, j) \in N1 \times N2 | (i, j) \notin N2\}
\]

\[
M02 = \{(i, j) \in N1 \times N2 | (i, j) \notin N1 \wedge (i, j) \notin N2\}
\]

They describe the number of links that overlap between the graphs (M12) and those that are present in one graph but not in the other (M10 and M02). The modified Jaccard index (Jm(N1, N2)) is defined as:

\[
Jm(N1, N2) = \frac{\#M12 \sum_{(i, j) \in M12} (M1_{ij} + M2_{ij})}{\#M12 \sum_{(i, j) \in M12} (M1_{ij} + M2_{ij}) + \#M10 \sum_{(i, j) \in M10} (M1_{ij} + M2_{ij}) + \#M02 \sum_{(i, j) \in M02} (M1_{ij} + M2_{ij})}
\]

The number of linkages within a given group (M12, M10 or M02) is weighted by the sum of exposures related to that links reflecting the importance of a given group of links.

Appendix 3. Covariance of the funding risk X

Default probabilities pj are modelled as random variables with variance estimated based on a time series of CDS spreads. Let (Ω, ℱ, P) be a probability space containing correlated random variables pj taking values from [0,1] and independent (pairwise and with all pj’s) random variables uj uniformly distributed on (0,1]. Let us define the σ-field generated by pj’s, i.e.

\[
\mathcal{F}_P = \sigma \left( \left\{ p_j^{-1}(C) \mid j \in \mathbb{N} \wedge C \in \mathcal{B}(0,1) \right\} \right),
\]

where \(\mathcal{B}(0,1)\) is a σ-field of Borel set on [0,1]. We use throughout the derivation the following fact:
For two given random variables \( Y_1 \) and \( Y_2 \), a \( \sigma \)-field \( \mathcal{G} \), such that \( Y_1 \) is \( \mathcal{G} \)-measurable and \( Y_2 \) is independent of \( \mathcal{G} \) and an integrable function \( f \),
\[
E[f(Y_1, Y_2)|\mathcal{G}] = E[f(y, Y_2)]|_{y=Y_1}
\]
The proof of the fact is standard: first for the elementary functions, then for their linear combinations and finally in the closure of the resulting subspace in \( L^1(\mathcal{F}) \).
Additionally, \( \tilde{X}_j = E[X_j|\mathcal{F}^p] = 1 - p_j \)
since \( \mathcal{F}^p \) is independent of \( u_j \).
\begin{align*}
\text{Case } i &\neq j \\
\text{cov}(X_i, X_j) &= E[(X_i - E(X_i))(X_j - E(X_j))] = E[E[X_iX_j - E(X_i)E(X_j)|\mathcal{F}^p]] \\
&= E[E[I_{u_i > \hat{p}_i}I_{u_j > \hat{p}_j}]|_{(\hat{p}_i, \hat{p}_j) = (p_i, p_j)}] \\
&
- (1 - E(p_i))(1 - E(p_j)) \quad \text{(C1)}
\end{align*}
the last equality once again following independence of \( u_i, u_j \) and \( \mathcal{F}^p \). Then, following independence of \( u_i \) and \( u_j \)
\[
E(1 - p_i)(1 - p_j) - (1 - E(p_i))(1 - E(p_j)) \\
= E(p_i p_j - E(p_i)E(p_j) = cov(p_i, p_j) \quad \text{(C2)}
\]
\begin{align*}
\text{Case } i &= j \\
\text{cov}(X_i, X_i) &= E[(I_{u_i > p_i})^2] - (1 - E(p_i))^2 \\
\text{Since } I_{u_i > p_i} \text{ takes only values 0 and 1, the square can simply be omitted in the first term under the expectation operator yielding} \\
&= (1 - E(p_i)) - (1 - E(p_i))^2 = E(p_i) - (E(p_i))^2 \quad \text{(C3)}
\end{align*}